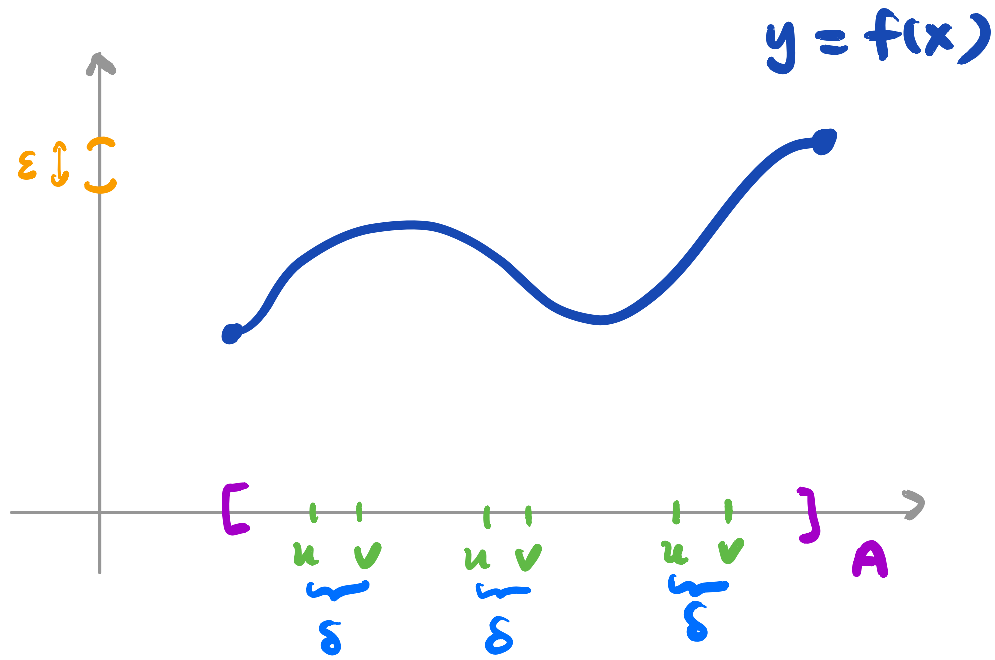
MATH 2050C Lecture 23 (Apr 14)

[Last Problem Set 12 posted, due on Apr 22.]

Last time: "Uniform" Continuity

Def^{n}: $f: A \to \mathbb{R}$ is uniformly continuous iff $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(u) - f(v)| < \varepsilon \quad \text{when } u, v \in A, \ |u - v| < \delta$$



Example: $f(x) = x$, $x \in \mathbb{R}$ unif. cts

Non-Examples: $f(x) = \frac{1}{x}$, $f(x) = \sin\frac{1}{x}$, $x \in (0,1)$

NOT unif. cts

Uniform Continuity Thm

$f : [a,b] \to \mathbb{R}$ cts everywhere $\Longrightarrow$ $f : [a,b] \to \mathbb{R}$ unif. cts

Example: $f(x) = x^2$ is unif. cts on $[-1,1]$.

We can also check directly from def^n.

Let $\varepsilon > 0$ be fixed but arbitrary.

Choose $\delta = \frac{\varepsilon}{4} > 0$. Then

whenever $u, v \in [-1,1]$

and $|u - v| < \delta$,

we have

$$|f(u) - f(v)|$$

$$= |u^2 - v^2|$$

$$\leq \underbrace{(|u| + |v|)}_{\because\, u,v \in [-1,1]} |u - v|$$

$$< 2\delta < \varepsilon$$

$u, v \in [-1, 1]$

& $|u-v| < \delta$

$\Downarrow$

$|f(u) - f(v)| < \varepsilon$

ie $|u^2 - v^2|$

$= |u+v| \cdot |u-v|$

$\leq (|u| + |v|) \cdot |u-v|$

$\leq (1+1) \cdot |u-v|$

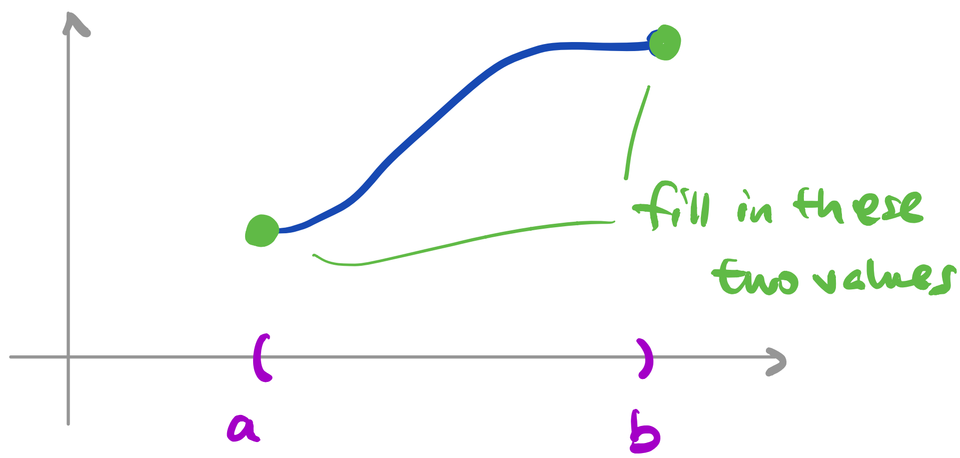
$= 2\delta < \varepsilon$

$\square$

Remark: $f(x) = x^2$ is NOT unif. cts on $\mathbb{R}$

Continuous Extension Thm

If $f : (a,b) \to \mathbb{R}$ is uniformly cts on (a,b),
then $\exists$ cts extension $\bar{f} : [a,b] \to \mathbb{R}$.



WANT: $\bar{f}(a) = \lim_{x \to a} f(x)$

$\bar{f}(b) = \lim_{x \to b} f(x)$

Q: why do the limit exist ?

Lemma: Suppose $f : A \to \mathbb{R}$ is uniformly cts.

(x_n) in A Cauchy seq. $\Longrightarrow$ $(f(x_n))$ in $\mathbb{R}$ Cauchy seq.

Proof: last lecture.

Proof of Continuous Extension Thm:

It suffices to show that the limits

$$\lim_{x\to a} f(x) \quad \& \quad \lim_{x\to b} f(x) \quad \text{exist.}$$

Then, the extension $\bar{f} : [a,b] \to \mathbb{R}$ defined by

$$\bar{f}(x) := \begin{cases} f(x), & x \in (a,b) \\ \lim_{x\to a} f(x), & x = a \\ \lim_{x\to b} f(x), & x = b \end{cases}$$

Ex: Check this is indeed cts on $[a,b]$.

Claim: $\lim_{x\to a} f(x)$ exists

Pf of Claim: By Sequential Criteria, we need to find some $L \in \mathbb{R}$ s.t.

$$\forall \text{ seq. } (x_n) \text{ in } (a,b) \text{ s.t. } \begin{cases} x_n \neq a & \forall n \in \mathbb{N} \\ \lim (x_n) = a \end{cases}$$

we have $(f(x_n)) \to L$

Step 1: Find ONE such L.

Choose $x_n := a + \frac{1}{n} \in (a,b)$ when n is large

Note: $(x_n) \to a$, hence is Cauchy

Lemma $\Rightarrow$ $(f(x_n))$ is also Cauchy

i.e. must be convergent.

say $\lim (f(x_n)) = L$.

Step 2: Show that the limit L obtained in Step 1 is "unique", i.e. does NOT depend on the choice of the seq (x_n).

Suppose (x_n') is any other seq. in (a,b)

s.t. $x_n' \neq a$ $\forall n \in \mathbb{N}$ & $\lim (x_n') = a$

Argument in Step 1 $\Rightarrow$

$\lim (f(x_n')) = L'$ for some $L' \in \mathbb{R}$.

We want to show $L' = L$.

Let $\varepsilon > 0$ be fixed but arbitrary.

By uniform continuity of f on (a,b),

$\exists\ \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(u) - f(v)| < \varepsilon \text{ when } |u - v| < \delta,\ u, v \in (a,b)$$

Idea: take $u = x_n$, $v = x_n'$

Since $\lim(x_n) = a = \lim(x_n')$,

$$\Rightarrow \quad \lim |x_n - x_n'| = 0$$

i.e. $\exists\ N = N(\delta) \in \mathbb{N}$ st

$$|x_n - x_n'| < \delta \quad \forall n \geq N$$

Therefore, $\forall\ n \geq N$, we have

$$|f(x_n) - f(x_n')| < \varepsilon$$

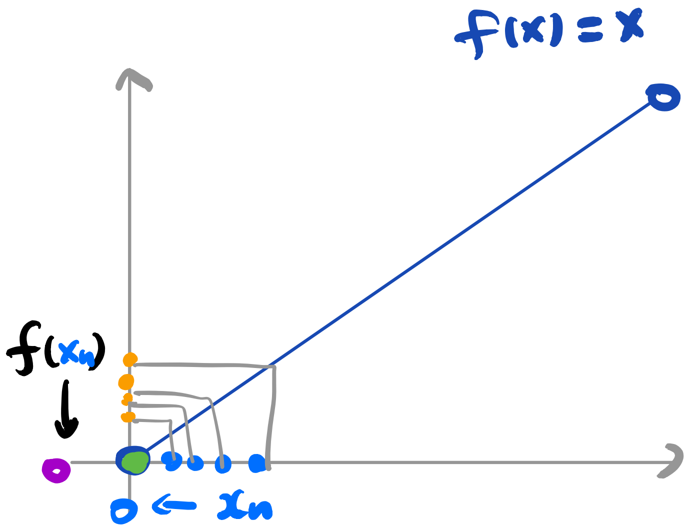
Take $n \to \infty$, by limit theorem

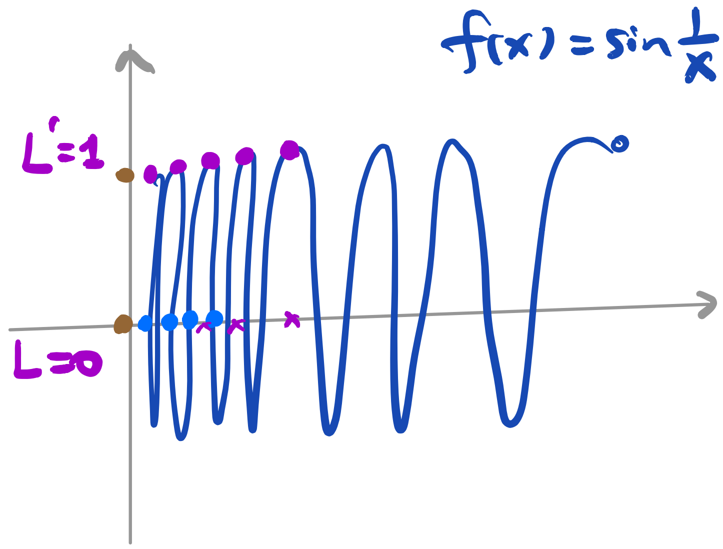
$$|L - L'| \leq \varepsilon$$

But $\varepsilon > 0$ is arbitrary, so $L = L'$.

$\square$

Picture illustrating the proof:





Final Exam up to here.